

Simple Skew Category Algebras Associated with Minimal Partially Defined Dynamical Systems

Patrik Lundström*

Johan Öinert†

In this article, we continue our study of category dynamical systems, that is functors s from a category G to Top^{op} , and their corresponding skew category algebras. Suppose that the spaces $s(e)$, for $e \in \text{ob}(G)$, are compact Hausdorff. We show that if (i) the skew category algebra is simple, then (ii) G is inverse connected, (iii) s is minimal and (iv) s is faithful. We also show that if G is a locally abelian groupoid, then (i) is equivalent to (ii), (iii) and (iv). Thereby, we generalize results by Öinert for skew group algebras to a large class of skew category algebras.

1 Introduction

Ever since the classical papers on ergodic theory and C^* -crossed products by Murray and von Neumann (see e.g. [15], [16] and [17]), we have known that there is a connection between topological properties of spaces and algebraical properties of rings. It has been observed by several authors that there is a link between freeness of topological dynamical systems and ideal intersection properties of C^* -algebras (see e.g. Zeller-Meier [40], Effros, Hahn [4], Elliott [5], Archbold, Quigg, Spielberg [1, 31, 32], Kawamura, Kishimoto, Tomiyama [7, 8, 38]). A lot of attention has also been given the connection between minimality of topological dynamical systems and simplicity of the corresponding C^* -algebras (see e.g. [3], [30], [37], [38] and [39]). To be more precise, suppose that X is a topological space and $s : X \rightarrow X$ is a continuous function; in that case the pair (X, s) is called a *topological dynamical system*. A subset Y of X is called *invariant* if $s(Y) \subseteq Y$. The topological dynamical system (X, s) is called *minimal* if there is no invariant closed nonempty proper subset of X . An element $x \in X$ is called *periodic* if there is a positive integer n such that $s^n(x) = x$; an element of X which is not

*Address: University West, Department of Engineering Science, SE-46186 Trollhättan, Sweden, E-mail: Patrik.Lundstrom@hv.se

†Address: Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark, E-mail: oinert@math.ku.dk

1 Introduction

periodic is called *aperiodic*. Recall that the topological dynamical system (X, s) is called *topologically free* if the set of aperiodic elements of X is dense in X . Now suppose that s is a homeomorphism of a compact Hausdorff space X . Denote by $C(X)$ the unital C^* -algebra of continuous complex-valued functions on X endowed with the supremum norm, pointwise addition and multiplication, and pointwise conjugation as involution. The map $\sigma_s : C(X) \rightarrow C(X)$ which to a function $f \in C(X)$ associates $f \circ s \in C(X)$ is then an automorphism of $C(X)$. The action of σ_s on $C(X)$ extends in a unique way to a strongly continuous representation $\sigma : \mathbb{Z} \rightarrow \text{Aut}(C(X))$ subject to the condition that $\sigma(1) = \sigma_s$, namely $\sigma(k) = \sigma_s^k$, for $k \in \mathbb{Z}$. In that case, the associated transformation group C^* -algebra $C^*(X, s)$ can be constructed (see e.g. [2] or [29] for the details). In 1978 Power showed the following elegant result connecting simplicity of $C^*(X, s)$ to minimality of s .

Theorem 1 (Power [30]). *If s is a homeomorphism of a compact Hausdorff space X of infinite cardinality, then $C^*(X, s)$ is simple if and only if (X, s) is minimal.*

Inspired by Theorem 1, the second author of the current article has in [18] and [27] shown analogous results (see Theorem 2) relating properties of an arbitrary topological dynamical system (X, s) , where s is a group homomorphism from a group G to $\text{Aut}(X)$, to ideal properties of the skew group algebra $C(X) \rtimes^\sigma G$. For more details concerning skew group algebras, and, more generally, skew category algebras, see Section 2.

Theorem 2 (Öinert [18] and [27]). *Suppose that X is a compact Hausdorff space and $s : G \rightarrow \text{Aut}(X)$ is a group homomorphism. Consider the following three assertions:*

- (i) $C(X) \rtimes^\sigma G$ is simple;
- (ii) s is minimal;
- (iii) s is faithful.

The following conclusions hold:

- (a) (i) implies (ii) and (iii);
- (b) if G is an abelian group, then (i) holds if and only if (ii) and (iii) hold;
- (c) if G is isomorphic to the additive group of integers and X has infinite cardinality, then (i) holds if and only if (ii) holds.

For related results concerning the ideal structure in skew group algebras and, more generally, group graded rings and Ore extensions, see [22], [23] [24], [25], [26], [33], [34] and [35].

A natural question is if there is a version of Theorem 2 that holds for dynamics defined by families of *partial* functions on a space, that is functions that do not necessarily have the same domain or codomain. In this article we address this question by using the machinery developed by the authors in [11] for *category dynamical systems*. These are defined by families, stable under composition, of continuous maps between potentially

1 Introduction

different topological spaces. We show a generalization of Theorem 2 to the case of skew category algebras defined by these maps and spaces (see Theorem 3 below). To be more precise, suppose that G is a category. The family of objects of G is denoted by $\text{ob}(G)$; we will often identify an object in G with its associated identity morphism. The family of morphisms in G is denoted by $\text{mor}(G)$; by abuse of notation, we will often write $n \in G$ when we mean $n \in \text{mor}(G)$. Throughout the article G is assumed to be small, that is with the property that $\text{mor}(G)$ is a set. The domain and codomain of a morphism n in G is denoted by $d(n)$ and $c(n)$ respectively. We let $G^{(2)}$ denote the collection of composable pairs of morphisms in G , that is all (m, n) in $\text{mor}(G) \times \text{mor}(G)$ satisfying $d(m) = c(n)$. For each $e \in \text{ob}(G)$, let G_e denote the collection of $n \in \text{mor}(G)$ with $d(n) = c(n) = e$. Let G^{op} denote the opposite category of G . We let Top denote the category having topological spaces as objects and continuous functions as morphisms. Suppose that $s : G \rightarrow \text{Top}^{\text{op}}$ is a (covariant) functor; in that case we say that s is a *category dynamical system*. If G is a groupoid, that is a category where all morphisms are isomorphisms, then we say that s is a *groupoid dynamical system*¹. If $e \in \text{ob}(G)$, then we say that an element $x \in s(e)$ is *periodic* if there is a nonidentity $n : e \rightarrow e$ in G such that $s(n)(x) = x$; an element of $s(e)$ which is not periodic is called *aperiodic*. We say that s is *topologically free* if for each $e \in \text{ob}(G)$, the set of aperiodic elements of $s(e)$ is dense in $s(e)$. We say that s is *minimal* if for each $e \in \text{ob}(G)$, there is no nonempty proper closed subset Y of $s(e)$ such that $s(n)(Y) \subseteq Y$ for all $n \in G_e$. We say that s is *faithful* if for each $e \in \text{ob}(G)$, and each nonidentity $n \in G_e$, there is $x \in s(e)$ such that $s(n)(x) \neq x$. For each $e \in \text{ob}(G)$, we let $C(e)$ denote the set of continuous complex valued functions on $s(e)$. For each $n \in G$ the functor s induces a map $\sigma(n) : C(d(n)) \rightarrow C(c(n))$ by the relation $\sigma(n)(f) = f \circ s(n)$, for $f \in C(d(n))$. If we use the terminology introduced in Section 2, then the map σ defines a *skew category system* (see Definition 4). In the same section, we show how one to each skew category system may associate a so called *skew category algebra* $A \rtimes^\sigma G$ (see also [19] for a more general construction) where $A = \bigoplus_{e \in \text{ob}(G)} C(e)$. In Section 2, we obtain two results concerning simplicity of a general skew category algebra $A \rtimes^\sigma G$. (see Proposition 11 and Proposition 14). These results are applied to category dynamical systems in Section 3, where we show the following generalization of Theorem 2 from groups to categories.

Theorem 3. *Let $s : G \rightarrow \text{Top}^{\text{op}}$ be a category dynamical system with the property that for each $e \in \text{ob}(G)$, the space $s(e)$ is compact Hausdorff. Consider the following four assertions:*

- (i) $\left[\bigoplus_{e \in \text{ob}(G)} C(e) \right] \rtimes^\sigma G$ is simple;
- (ii) G is inverse connected;
- (iii) s is minimal;
- (iv) s is faithful.

¹The notion *groupoid dynamical system* is in the C^* -algebra literature used in a sense different from ours, see e.g. [12] and [13].

2 Simple Skew Category Algebras

The following conclusions hold:

- (a) (i) implies (ii), (iii) and (iv);
- (b) if G is a locally abelian groupoid, then (i) holds if and only if (ii), (iii) and (iv) hold;
- (c) if G is a groupoid with the property that for each $e \in \text{ob}(G)$ the space $s(e)$ is infinite and the group G_e is isomorphic to \mathbb{Z} , then (i) holds if and only if (ii) and (iii) hold.

Various types of crossed product algebras associated to groupoid dynamical systems, and even more general types of dynamical systems, have appeared in the literature before (see e.g. [6, 12, 13, 28]). The difference between these algebras and our algebras is that, generally speaking, our skew category algebras are defined in an algebraic way, without making use of any topology. For example, by choosing A to be a C^* -algebra and our category G to be a locally compact group acting on A , we may form the skew category algebra $A \rtimes^\sigma G$, and also the standard crossed product C^* -algebra appearing in e.g. [36]. The relation between these two algebras is that the skew category algebra sits as a dense subalgebra inside the crossed product C^* -algebra.

2 Simple Skew Category Algebras

In this section, we first recall the definitions of skew category systems (A, G, σ) and skew category algebras $A \rtimes^\sigma G$ from [11] (see Definition 4 and Definition 6). Thereafter, we show two results concerning simplicity of skew category algebras and properties of skew category systems (see Proposition 11 and Proposition 14). These results will be applied to category dynamical systems in Section 3.

Conventions on rings. Let R be an associative ring. The identity map $R \rightarrow R$ is denoted by id_R . If R is unital then the identity element of R is nonzero and is denoted by 1_R . The category of unital rings is denoted by Ring . We say that a subset R' of R is a subring of R if it is itself a ring under the binary operations of R . We always assume that ring homomorphisms between unital rings respect the identity elements. If A is a subset of R , then the *commutant* of A in R is the set of elements of R that commute with every element of A . If A is a commutative subring of R , then A is called *maximal commutative* in R if the commutant of A in R equals A . All ideals of rings are supposed to be two-sided. By a *nontrivial ideal* we mean a proper nonzero ideal. If R is commutative and $x \in R$, then $\text{Ann}(x)$ denotes the ideal of R consisting of all $y \in R$ satisfying $xy = 0$. If G is a monoid of endomorphisms of a ring A , then we say that a subset B of A is *G -invariant* if for every $g \in G$ the inclusion $g(B) \subseteq B$ holds. The ring A is called *G -simple* if there is no nontrivial G -invariant ideal of A .

Conventions on categories. Let G be a category. Recall that G is called *connected* if its underlying undirected graph is connected. Note that if G is a groupoid, then G is connected precisely when there to each pair $e, f \in \text{ob}(G)$, is $n \in \text{mor}(G)$ with $d(n) = e$ and $c(n) = f$. We say that G is *locally a group* (*locally abelian*) if each

2 Simple Skew Category Algebras

monoid G_e , for $e \in \text{ob}(G)$, is a group (abelian). We say that G is *inverse connected* if given $e, f \in \text{ob}(G)$, there are $m, n \in \text{mor}(G)$ with $d(m) = c(n) = f$ and $mn = e$. Note that if G is both inverse connected and locally a group, then G is a groupoid. A congruence relation R on G is a collection of equivalence relations $R_{e,f}$ on $\text{hom}(e, f)$, for $e, f \in \text{ob}(G)$, chosen so that if $(m, m') \in R_{e,f}$ and $(n, n') \in R_{f,g}$, then $(mn, m'n') \in R_{e,g}$, for all $e, f, g \in \text{ob}(G)$. If $e, f \in \text{ob}(G)$ and $n \in \text{hom}(e, f)$, then we let $[n]$ denote the equivalence class in $\text{hom}(e, f)$ defined by $R_{e,f}$. Suppose that H is another category and that $F : G \rightarrow H$ is a functor. The kernel of F , denoted $\ker(F)$, is the congruence relation on G defined by letting $(m, n) \in \ker(F)_{e,f}$, for $e, f \in \text{ob}(G)$, whenever $m, n \in \text{hom}(e, f)$ and $F(m) = F(n)$. Recall that $\ker(F)$ is called *trivial* if for each $e, f \in \text{ob}(G)$, $\ker(F)_{e,f}$ is the equality relation on $\text{hom}(e, f)$. We say that $\ker(F)$ is *locally trivial* if for each $e \in \text{ob}(G)$, $\ker(F)_{e,e}$ is the equality relation on $\text{hom}(e, e)$.

Definition 4. By a *skew category system* we mean a triple (A, G, σ) where G is a (small) category, A is the direct sum of unital rings A_e , for $e \in \text{ob}(G)$, and σ is a functor $G \rightarrow \text{Ring}$ satisfying $\sigma(n) : A_{d(n)} \rightarrow A_{c(n)}$, for $n \in G$.

Remark 5. Suppose that (A, G, σ) is a skew category system. The fact that σ is a functor $G \rightarrow \text{Ring}$ can be formulated in terms of maps by saying that

$$\sigma(e) = \text{id}_{A_e} \quad (1)$$

for all $e \in \text{ob}(G)$, and

$$\sigma(m)\sigma(n) = \sigma(mn) \quad (2)$$

for all $(m, n) \in G^{(2)}$.

Definition 6. If (A, G, σ) is a skew category system, then we let $A \rtimes^\sigma G$ denote the collection of formal sums $\sum_{n \in G} a_n u_n$, where $a_n \in A_{c(n)}$, $n \in G$, are chosen so that all but finitely many of them are nonzero. Define addition and multiplication on $A \rtimes^\sigma G$ by

$$\left(\sum_{n \in G} a_n u_n \right) + \left(\sum_{n \in G} b_n u_n \right) = \sum_{n \in G} (a_n + b_n) u_n \quad (3)$$

respectively

$$\left(\sum_{n \in G} a_n u_n \right) \left(\sum_{n \in G} b_n u_n \right) = \sum_{n \in G} \left(\sum_{\substack{(m, m') \in G^{(2)}; \\ mm' = n}} a_m \sigma(m)(b_{m'}) \right) u_n \quad (4)$$

for $\sum_{n \in G} a_n u_n, \sum_{n \in G} b_n u_n \in A \rtimes^\sigma G$. It is clear that these operations define a ring structure on $A \rtimes^\sigma G$. We call $A \rtimes^\sigma G$ the *skew category algebra* defined by (A, G, σ) . Often we let u_n denote $1_{A_{c(n)}} u_n$ for all $n \in G$.

2 Simple Skew Category Algebras

Remark 7. If G is a groupoid, then (4) can be rewritten in the following slightly simpler form

$$\left(\sum_{n \in G} a_n u_n \right) \left(\sum_{n \in G} b_n u_n \right) = \sum_{n \in G} \left(\sum_{\substack{m \in G; \\ c(m)=c(n)}} a_m \sigma(m) (b_{m^{-1}n}) \right) u_n,$$

which, in the case when G is a group, simplifies even more to

$$\left(\sum_{n \in G} a_n u_n \right) \left(\sum_{n \in G} b_n u_n \right) = \sum_{n \in G} \left(\sum_{m \in G} a_m \sigma(m) (b_{m^{-1}n}) \right) u_n.$$

Remark 8. Suppose that $T := A \rtimes^\sigma G$ is a skew category algebra. If we for each $n \in G$, put $T_n = A_{c(n)} u_n$, then $T = \bigoplus_{n \in G} T_n$, $T_m T_n = T_{mn}$, for $(m, n) \in G^{(2)}$, and $T_m T_n = \{0\}$, otherwise. In the terminology of [9], [10], [20] and [21] this means that a skew category algebra is a strongly category graded ring.

Proposition 9. *If $A \rtimes^\sigma G$ is a skew category algebra and I is an ideal of $A \rtimes^\sigma G$, then*

- (a) *the equality $I = A \rtimes^\sigma G$ holds if and only if the equality $I \cap A_e = A_e$ holds for all $e \in \text{ob}(G)$;*
- (b) *if G is inverse connected, then the equality $I = A \rtimes^\sigma G$ holds if and only if the equality $I \cap A_e = A_e$ holds for some $e \in \text{ob}(G)$.*

Proof. Let R denote $A \rtimes^\sigma G$.

(a) The "only if" statement is clear. Now we show the "if" statement. Suppose that for every $e \in \text{ob}(G)$ the equality $I \cap A_e = A_e$ holds. Take $x \in R$. From the definition of skew category algebras it follows that there is a finite subset X of $\text{ob}(G)$ satisfying

$$x \left(\sum_{e \in X} u_e \right) = x. \tag{5}$$

But $\sum_{e \in X} u_e \in \sum_{e \in X} A_e = \sum_{e \in X} I \cap A_e \subseteq I$ which, by equation (5), implies that $x \in I$. Since x was arbitrarily chosen from R , we get that $I = R$.

(b) The "only if" statement is clear. Now we show the "if" statement. Suppose that $I \cap A_e = A_e$ for some $e \in \text{ob}(G)$. Take $f \in \text{ob}(G)$. By (a) we are done if we can show that $I \cap A_f = A_f$. Since G is inverse connected, there are $m, n \in G$ with $d(m) = c(n) = e$ such that $f = mn$. But then $u_f = u_m u_e u_n \in I \cap A_f$. This implies that $I \cap A_f = A_f$. \square

Proposition 10. *Let $A \rtimes^\sigma G$ be a skew category algebra. Suppose that R is a congruence relation contained in $\ker(\sigma)$. If I is the two-sided ideal in $A \rtimes^\sigma G$ generated by an element $\sum_{n \in \text{mor}(G)} a_n u_n$, where $a_n \in A_{c(n)}$, for $n \in \text{mor}(G)$, with $a_n = 0$ for all but finitely many $n \in \text{mor}(G)$, satisfying $a_n = 0$ if n does not belong to any of the classes $[e]$, for $e \in \text{ob}(G)$, and $\sum_{n \in [e]} a_n = 0$, for $e \in \text{ob}(G)$, then $A \cap I = \{0\}$. In particular, if $A \rtimes^\sigma G$ is simple, then $\ker(\sigma)$ is locally trivial.*

2 Simple Skew Category Algebras

Proof. See the proof of [19, Proposition 9]. □

Proposition 11. *Let $A \rtimes^\sigma G$ be a skew category algebra. Consider the following five assertions:*

- (i) $A \rtimes^\sigma G$ is simple;
- (ii) G is inverse connected;
- (iii) for each $e \in \text{ob}(G)$, the ring A_e is G_e -simple;
- (iv) for each $e \in \text{ob}(G)$, the ring $Z(A_e \rtimes^\sigma G_e)$ is a field;
- (v) $\ker(\sigma)$ is locally trivial.

The following conclusions hold:

- (a) (i) implies (ii)-(v);
- (b) if G is a locally abelian groupoid, then (i) holds if and only if (ii)-(v) hold.

Proof. Let R denote $A \rtimes^\sigma G$.

(a) Suppose that (i) holds. We first show (ii). Take $e, f \in \text{ob}(G)$. Let I denote the ideal generated by u_e in R . Since R is simple it follows that $I = R$. In particular, we get that $u_f \in I$. Since I consists of the set of finite sums of the form $xu_e y$, where $x, y \in R$, it follows that there exist $m, n \in \text{mor}(G)$ with $d(m) = c(n) = e$ such that $u_f = u_m u_e u_n$. This implies that $f = mn$ and hence that G is inverse connected. Now we show (iii). Take $e \in \text{ob}(G)$ and a nonzero G_e -invariant ideal J_e of A_e . Let I denote the ideal of R generated by $J_e u_e$. Since R is simple we get that $I = R$. This implies in particular that $u_e \in I$. Then $u_e \in I \cap A_e u_e = J_e u_e$ which implies that $1_{A_e} \in J_e$. Hence $J_e = A_e$. Now we show (iv). Let $e \in \text{ob}(G)$. Take a nonzero x in $Z(A_e \rtimes^\sigma G_e)$ and let I be the ideal of R generated by x . Since I is nonzero and R is simple, we get that $I = R$. In particular, u_e equals a finite sum of elements of the form $y x z$ where $y, z \in A_e \rtimes^\sigma G_e$. But since x belongs to $Z(A_e \rtimes^\sigma G_e)$ we get that $u_e = w x = x w$ for some $w \in A_e \rtimes^\sigma G_e$. All that is left to show now is that $w \in Z(A_e \rtimes^\sigma G_e)$. Take $v \in A_e \rtimes^\sigma G_e$. Then, since x commutes with v , we get that $w v = v w u_e = v w x w = w x v w = u_e v w = v w$. Assertion (v) follows immediately from Proposition 10.

(b) From (a) it follows that we only need to show the "if" statement. Suppose that (i)-(v) hold. Let I be a nonzero ideal of R . Then there is a nonzero element $x = \sum_{n \in \text{mor}(G)} a_n u_n$ in I , where $a_n \in A_{c(n)}$, for $n \in \text{mor}(G)$ and $a_n = 0$ for all but finitely many $n \in \text{mor}(G)$, with the property that $a_e \neq 0$ for some $e \in \text{ob}(G)$. Indeed, take a nonzero $y = \sum_{m \in \text{mor}(G)} b_m u_m$ in I . We now consider two cases. Case 1: There is $n \in \text{mor}(G)$ with $d(n) = c(n)$ and $b_n \neq 0$. Then $x = y u_{n^{-1}}$ has the desired property where $e = c(n)$. Case 2: There is $n \in \text{mor}(G)$ with $d(n) \neq c(n)$ and $b_n \neq 0$. Since G is inverse connected it follows that there are $m, p \in \text{mor}(G)$ with $d(m) = c(n)$, $c(p) = d(p)$ and $mp = c(n)$. Then $x = y u_{(nm)^{-1}}$ has the desired property where $e = c(n)$. Let J be the ideal of $A_e \rtimes^\sigma G_e$ consisting of all $b \in A_e$ such that there are $b_n \in A_e$, for $n \in G_e \setminus \{e\}$,

2 Simple Skew Category Algebras

with the property that $b_n \in A_e$, for $n \in G_e \setminus \{e\}$, where $b_n = 0$ for all but finitely many $n \in G_e \setminus \{e\}$, and $y := b + \sum_{n \in G_e \setminus \{e\}} b_n u_n \in u_e I u_e$ and $\text{Supp}(y) \subseteq \text{Supp}(x)$. By the above discussion concerning the element x it follows that J is nonzero. Now we show that J is G_e -invariant. Take $m \in G_e$. Then, since G_e is abelian, it follows that

$$I \ni u_m y u_{m^{-1}} = u_m b u_{m^{-1}} + \sum_{n \in G_e \setminus \{e\}} u_m b_n u_n u_{m^{-1}} = \sigma(m)(b) + \sum_{n \in G_e \setminus \{e\}} \sigma(m)(b_n) u_n.$$

Therefore, $\sigma(m)(b) \in J$. By G_e -simplicity of A_e it follows that $J = A_e$. In particular, we can choose y so that $b = 1_{A_e}$. Among all nonzero elements $z = \sum_{n \in G_e} c_n u_n \in u_e I u_e$, with $c_n = 0$ for all but finitely many $n \in G_e$, choose an element minimizing $|\text{Supp}(z)|$. By the above discussion, we can assume that $c_e = 1$ for such an element z . Now we show that $z \in Z(A_e \rtimes^\sigma G_e)$. Take $a \in A_e$ and $m \in G_e$. Then, since G_e is abelian, it follows that

$$a u_m z - z a u_m = \sum_{n \in G_e} (a \sigma(m)(c_n) - c_n \sigma(n)(a)) u_{mn}.$$

Since

$$a \sigma(m)(c_e) - c_e \sigma(e)(a) = a \sigma(m)(1_{A_e}) - 1_{A_e} a = a 1_{A_e} - 1_{A_e} a = a - a = 0$$

we get that $|\text{Supp}(a u_m z - z a u_m)| < |\text{Supp}(z)|$. Since $a u_m z - z a u_m \in u_e I u_e$, we get, by minimality of $|\text{Supp}(z)|$, that $a u_m z - z a u_m = 0$. Therefore $z \in Z(A_e \rtimes^\sigma G_e)$. Since $Z(A_e \rtimes^\sigma G_e)$ is a field and z is nonzero, it follows that z is invertible in $A_e \rtimes^\sigma G_e$ and hence that $u_e I u_e = A_e \rtimes^\sigma G_e$. In particular $I \cap A_e = A_e$. Simplicity of $A \rtimes^\sigma G$ now follows directly from Proposition 9(b). \square

Proposition 12. *Let $A \rtimes^\sigma G$ be a skew category algebra with A commutative. Consider the following two assertions:*

- (i) *if I is a nonzero ideal of $A \rtimes^\sigma G$, then $I \cap A \neq \{0\}$;*
- (ii) *the subring A is maximal commutative in $A \rtimes^\sigma G$.*

The following conclusions hold:

- (a) (i) *implies* (ii);
- (b) (ii) *does not imply* (i) *for all categories* G ;
- (c) *if G is a groupoid, then (i) holds if and only if (ii) holds.*

Proof. See the proof of [11, Proposition 2.3]. \square

Remark 13. For other results related to the implication (i) implies (ii) in Proposition 12, see [19]. The implication (ii) implies (i) in Proposition 12 actually holds for all nondegenerate groupoid graded rings, see [20].

Proposition 14. *Let $A \rtimes^\sigma G$ be a skew category algebra with A commutative. Consider the following six assertions:*

3 Minimal and Faithful Category Dynamical Systems

- (i) $A \rtimes^\sigma G$ is simple;
- (ii) G is inverse connected;
- (iii) for each $e \in \text{ob}(G)$, the ring A_e is G_e -simple;
- (iv) for each $e \in \text{ob}(G)$, the ring $Z(A_e \rtimes^\sigma G_e)$ is a field;
- (v) $\ker(\sigma)$ is locally trivial.
- (vi) A is maximal commutative in $A \rtimes^\sigma G$.

The following conclusions hold:

- (a) (i) implies (ii)-(vi);
- (b) if G is a groupoid, then (i) holds if and only if (ii)-(vi) hold.

Proof. Let R denote $A \rtimes^\sigma G$.

(a) Suppose that (i) holds. By Proposition 11(a) we get that (ii)-(v) hold. It follows from Proposition 12(a) that (vi) holds.

(b) Suppose that G is a groupoid. By (a) we only need to show the "if" statement. Suppose that (ii)-(vi) hold. We show (i). Let I be a nonzero ideal of R . By Proposition 12(b), we get that $I \cap A$ is a nonzero ideal of A . Since $I \cap A = \sum_{e \in \text{ob}(G)} I \cap A_e$, there is $e \in \text{ob}(G)$ such that $I \cap A_e$ is a nonzero ideal of A_e . Take $n \in G_e$. From the fact that G is a groupoid, we get that there is $m \in G_e$ such that $nm = e$. Then

$$\begin{aligned} n(I \cap A_e)u_e &= \sigma(n)(I \cap A_e)u_{nm} = \sigma(n)(I \cap A_e)u_n u_m = u_n(I \cap A_e)u_m \subseteq \\ &\subseteq (u_n I u_m) \cap u_n A_e u_m \subseteq I \cap \sigma(n)(A_e)u_n u_m \subseteq (I \cap A_e)u_{nm} = (I \cap A_e)u_e. \end{aligned}$$

Hence $u_n(I \cap A_e) \subseteq I \cap A_e$ and thus $I \cap A_e$ is also G_e -invariant. By G_e -simplicity of A_e this implies that $I \cap A_e = A_e$. By Proposition 9(b), we get that $I = R$. \square

3 Minimal and Faithful Category Dynamical Systems

In this section, we prove Theorem 3. To this end, we need a result from [11] (see Proposition 15) concerning topological freeness and maximal commutativity. We shall also need three results (see Proposition 17, Proposition 19 and Proposition 20) relating minimality of category dynamical systems to simplicity of the corresponding skew category algebras. In the end of this section, we discuss the implications of these results for the connection between dynamical systems defined by partially defined functions (see Definition 24) and properties of the corresponding skew category algebras (see Examples 26 and 28).

Let $s : G \rightarrow \text{Top}^{\text{op}}$ be a category dynamical system. Then $(\oplus_{e \in \text{ob}(G)} C(e), G, \sigma)$ is a skew category system. Indeed, we need to check conditions (1) and (2) from Remark 5. Take $e \in \text{ob}(G)$ and $f \in C(e)$. Then $\sigma(e)(f) = f \circ s(e) = f$. Therefore $\sigma(e) = \text{id}_{C(e)}$. Take $(m, n) \in G^{(2)}$ and $f \in C(d(n))$. Then $\sigma(m)\sigma(n)(f) = \sigma(m)(f \circ s(n)) = f \circ s(n) \circ s(m) = f \circ (s(m) \circ_{\text{op}} s(n)) = f \circ s(mn) = \sigma(mn)(f)$. Therefore $\sigma(m)\sigma(n) = \sigma(mn)$. Hence, we may form the skew category algebra $(\oplus_{e \in \text{ob}(G)} C(e)) \rtimes^\sigma G$.

3 Minimal and Faithful Category Dynamical Systems

Proposition 15. *Suppose that $s : G \rightarrow \text{Top}^{\text{op}}$ is a groupoid dynamical system. If for each $e \in \text{ob}(G)$ the space $s(e)$ is locally compact Hausdorff, then the following two assertions are equivalent:*

- (i) *s is topologically free;*
- (ii) *the subring $\oplus_{e \in \text{ob}(G)} C(e)$ is maximal commutative in $[\oplus_{e \in \text{ob}(G)} C(e)] \rtimes^{\sigma} G$.*

Proof. See the proof of [11, Theorem 3.2]. □

Lemma 16. *Suppose that X and Y are topological spaces and $A \subseteq X$ and $B \subseteq Y$. If $f : X \rightarrow Y$ is a continuous function such that $f(A) \subseteq B$, then $f(\overline{A}) \subseteq \overline{B}$.*

Proof. The inclusion $f(A) \subseteq B$ can equivalently be stated as the inclusion $A \subseteq f^{-1}(B)$. Since f is continuous we get that $f^{-1}(\overline{B})$ is a closed subset of X containing A and hence also \overline{A} , i.e. $\overline{A} \subseteq f^{-1}(\overline{B})$ or equivalently $f(\overline{A}) \subseteq \overline{B}$. □

Proposition 17. *If $s : G \rightarrow \text{Top}^{\text{op}}$ is a minimal category dynamical system such that for each $e \in \text{ob}(G)$ the topological space $s(e)$ is infinite and Hausdorff and G_e is isomorphic to the additive group of integers, then s is topologically free and, hence, faithful.*

Proof. Take $e \in \text{ob}(G)$. We will show that every point in $s(e)$ is aperiodic, which, of course, implies that the set of aperiodic points in $s(e)$ is dense. Take $x \in s(e)$. Then $G_e(x)$ is a nonempty G_e -invariant subset of $s(e)$. By Lemma 16 the set $\overline{G_e(x)}$ is a nonempty closed G_e -invariant subset of X , which, since s is minimal, implies that $\overline{G_e(x)} = s(e)$. Since $s(e)$ is infinite and Hausdorff it follows that $G_e(x)$ is infinite. Let g be a generator for G_e . Seeking a contradiction, suppose that there is a nonzero integer N such that $s(g^N)(x) = x$. Then the cardinality of $G_e(x)$ is less than or equal to $|N|$. This contradicts the fact that $G_e(x)$ is an infinite set. Therefore, $s(g^N)(x) \neq x$, for all nonzero integers N , and hence x is an aperiodic point in $s(e)$. □

Recall that a topological space X is called *completely regular* if given any closed proper subset F of X , there is a nonzero continuous complex valued function on X that vanishes on F .

Lemma 18. *Every compact Hausdorff topological space is completely regular.*

Proof. See any standard book on point set topology, e.g. [14]. □

Proposition 19. *Suppose that $s : G \rightarrow \text{Top}^{\text{op}}$ is a category dynamical system with each $s(e)$, for $e \in \text{ob}(G)$, compact Hausdorff. If for each $e \in \text{ob}(G)$, the ring $C(e)$ is G_e -simple, then s is minimal.*

Proof. We show the contrapositive statement. Suppose that s is not minimal. Then there is $e \in \text{ob}(G)$ such that $s(e)$ is not G_e -minimal, that is there is a closed nonempty proper G_e -invariant subset Y of $s(e)$. Define I_Y to be the set of $f \in C(e)$ that vanish on Y . It is clear that I_Y is a G_e -invariant proper ideal of $C(e)$. By Lemma 18, it follows that I_Y is also nonzero. Therefore, $C(e)$ is not G_e -simple. □

Proposition 20. *Suppose that $s : G \rightarrow \text{Top}^{\text{op}}$ is a category dynamical system with each $s(e)$, for $e \in \text{ob}(G)$, compact. If s is minimal, then for each $e \in \text{ob}(G)$, the ring $C(e)$ is G_e -simple.*

Proof. We show the contrapositive statement. Suppose that there is $e \in \text{ob}(G)$ such that $C(e)$ is not G_e -simple. Then there is a nontrivial G_e -invariant ideal I of $C(e)$. For a subset J of $C(e)$, let N_J denote the set $\bigcap_{f \in J} f^{-1}(\{0\})$. We claim that N_I is a closed, nonempty proper G_e -invariant subset of $C(e)$. If we assume that the claim holds, then s is not minimal and the proof is done. Now we show the claim. Since I is G_e -invariant the same is true for N_I . Since I is nonzero it follows that N_I is a proper subset of $s(e)$. Since each set $f^{-1}(\{0\})$, for $f \in I$, is closed, the same is true for N_I . Seeking a contradiction, suppose that N_I is empty. Since $C(e)$ is compact, there is a finite subset J of I such that $N_J = N_I$. Then the function $F = \sum_{f \in J} |f|^2 = \sum_{f \in J} f \bar{f}$ belongs to I and, since N_J is empty, it has the property that $F(x) \neq 0$ for all $x \in s(e)$. Therefore $1_e = F \cdot \frac{1}{F} \in I$, where 1_e denotes the constant map $s(e) \rightarrow \mathbb{C}$ which sends each element of $s(e)$ to 1. This implies that $I = C(e)$ which is a contradiction. Therefore N_I is nonempty. \square

Proposition 21. *If $s : G \rightarrow \text{Top}^{\text{op}}$ is a category dynamical system with each $s(e)$, for $e \in \text{ob}(G)$, compact Hausdorff, then s is faithful if and only if $\ker(\sigma)$ is locally trivial.*

Proof. Suppose that s is not faithful. Then there is $e \in \text{ob}(G)$ and a nonidentity $n \in G_e$ such that $s(n) = \text{id}_{s(e)}$. This implies that $\sigma(n) = \text{id}_{C(e)}$ and hence that σ is not locally trivial.

Suppose that σ is not locally trivial. Then there is $e \in \text{ob}(G)$ and a nonidentity $n \in G_e$ such that $\sigma(n) = \text{id}_{C(e)}$. This implies that $f(s(n)(x)) = f(x)$ for all $f \in C(e)$ and all $x \in s(e)$. By Urysohn's lemma (see any standard book on point set topology, e.g. [14]), the set of continuous complex valued functions on a compact Hausdorff space separates points. Hence, we get that $s(n)(x) = x$ for all $x \in s(e)$. Therefore s is not faithful. \square

Proof of Theorem 3. Let R denote $[\oplus_{e \in \text{ob}(G)} C(e)] \rtimes^\sigma G$.

(a) Suppose that R is simple. By Proposition 11(a) it follows that G is inverse connected and that each $C(e)$, for $e \in \text{ob}(G)$, is G_e -simple. By Proposition 19, this implies that s is minimal. By Proposition 11(a) again it follows that $\ker(\sigma)$ is locally trivial. By Proposition 21 we get that s is faithful.

(b) By (a) we only need to show the "if" statement. Suppose that G is a locally abelian inverse connected groupoid and that s is minimal and faithful. We show that R is simple. By Proposition 20 it follows that each $C(e)$, for $e \in \text{ob}(G)$, is G_e -simple and by Proposition 21 we conclude that $\ker(\sigma)$ is locally trivial. Take $e \in \text{ob}(G)$. We claim that $Z(C(e) \rtimes^\sigma G_e)$ is a field. If we assume that the claim holds, then, by Proposition 11(b), it follows that R is simple. Now we show the claim. We will in fact show that $Z(C(e) \rtimes^\sigma G_e) = \mathbb{C}$ where we identify a complex number z with the constant function 1_z in $C(e)$ that maps each element of $s(e)$ to z . It is clear that $Z(C(e) \rtimes^\sigma G_e) \supseteq \mathbb{C}$. Now we show the inclusion $Z(C(e) \rtimes^\sigma G_e) \subseteq \mathbb{C}$. Take $\sum_{n \in G_e} f_n u_n \in Z(C(e) \rtimes^\sigma G_e)$ where $f_n \in C(e)$, for $n \in G(e)$, and $f_n = 0$ for all but finitely many $n \in G_e$. For every $m \in G_e$, the equality $u_m (\sum_{n \in G_e} f_n u_n) = (\sum_{n \in G_e} f_n u_n) u_m$ holds. From the fact that G_e is

abelian, we get that $f_n(s(m)(x)) = f_n(x)$, for $m, n \in G_e$ and $x \in s(e)$. For every $n \in G_e$ choose a complex number z_n in the image of f_n . Since $f_n \circ s(m) = f_n$ it follows that the set $f_n^{-1}(z_n)$ is a nonempty G_e -invariant closed subset of $s(e)$. Since s is minimal it follows that $f_n^{-1}(z_n) = s(e)$ and hence that $f_n = 1_{z_n}$, for $n \in G_e$. Take a nonidentity $m \in G_e$. From the fact that s is faithful, we get that there is $a \in s(e)$ such that $s(m)(a) \neq a$. Since $C(e)$ separates the points in $s(e)$ there is $g \in C(e)$ such that $g(a) \neq g(s(m)(a))$. Since $\sum_{n \in G_e} 1_{z_n} u_n$ commutes with g we get that $1_{z_m}(x)(g(x) - \sigma(m)(g)(x)) = 0$ for all $x \in s(e)$. By specializing this equality with $x = a$, we get that $z_m(g(a) - g(s(m)(a))) = 0$ which in turn implies that $z_m = 0$. Therefore, the inclusion $Z(C(e) \rtimes^\sigma G_e) \subseteq \mathbb{C}$ holds.

(c) We can show this in two ways. Either, we use Theorem 3(b) and the faithful part of Proposition 17, or we can construct a direct proof (similar to the proof of (b) above) using Proposition 14, the topologically free part of Proposition 17 and Proposition 15. \square

Remark 22. If we omit the condition that s is faithful, then the conditions (ii) and (iii) in Theorem 3 do not necessarily imply that $[\oplus_{e \in \text{ob}(G)} C(e)] \rtimes^\sigma G$ is simple. In fact, let G and H be any nontrivial groups. By abuse of notation, we let e denote the identity element of both groups. Suppose that X is a compact Hausdorff space equipped with a minimal G -action $G \times X \ni (g, x) \mapsto g(x) \in X$. Define an action of $G \times H$ on X by the relation $G \times H \times X \ni (g, h, x) \mapsto g(x) \in X$; this action is also minimal. Then $C(X) \rtimes^\sigma (G \times H)$ is not simple. In fact, let I be the ideal generated by the set of elements of the form $u_{(e,e)} - u_{(e,h)}$, for $h \in H$. Define the homomorphism of abelian groups $\varphi : C(X) \rtimes^\sigma (G \times H) \rightarrow C(X)$ by the additive extension of the relation $\varphi(f u_{(g,h)}) = f$, for $g \in G$, $h \in H$ and $f \in C(X)$. We claim that $I \subseteq \ker(\varphi)$. If we assume that the claim holds, then I is a nontrivial ideal of $C(X) \rtimes^\sigma (G \times H)$, since $\varphi|_{C(X)} = \text{id}_{C(X)}$. Now we show the claim. By the definition of I it follows that it is enough to show that φ maps elements of the form $f_1 u_{(r,s)}(u_{(e,e)} - u_{(e,h)}) f_2 u_{(t,v)}$ to zero, where $f_1, f_2 \in C(X)$, $r, t \in G$ and $s, h, v \in H$. However, since $\sigma(e, h)(f_2) = f_2$, we get that $f_1 u_{(r,s)}(u_{(e,e)} - u_{(e,h)}) f_2 u_{(t,v)} = f_1 \sigma(r, s)(f_2)(u_{(rt,sv)} - u_{(rt,shv)})$ which, obviously, is mapped to zero by φ .

Remark 23. If we omit the condition that G is locally abelian in Theorem 3, then the conditions (ii), (iii) and (iv) do not necessarily imply that $[\oplus_{e \in \text{ob}(G)} C(e)] \rtimes^\sigma G$ is simple. Indeed, Öinert has given an example of this phenomenon when G is the nonabelian group of homeomorphisms of the circle S^1 acting on the compact Hausdorff space S^1 (for the details, see [27, Example 6.1]).

Definition 24. Suppose that X is a topological space. By a *partially defined dynamical system* on X we mean a collection P of functions such that:

- if $f \in P$, then the domain $d(f)$ of f and the codomain $c(f)$ of f are subsets of X and f is continuous as a function $d(f) \rightarrow c(f)$ where $d(f)$ and $c(f)$ are equipped with the relative topologies induced by the topology on X ;
- if $f \in P$, then $\text{id}_{d(f)} \in P$ and $\text{id}_{c(f)} \in P$;

3 Minimal and Faithful Category Dynamical Systems

- if $f, g \in P$ are such that $d(f) = c(g)$, then $f \circ g \in P$.

We say that an element x of X is *periodic* with respect to P if there is a nonidentity function f in P with $d(f) = c(f)$ and $f(x) = x$. An element x is *aperiodic* with respect to P if it is not periodic. We say that P is *topologically free* if the set of aperiodic elements of X is dense in X . We say that P is *minimal* if for every $Y \subseteq X$, satisfying $Y = d(g)$ for some $g \in P$, there is no nonempty proper closed subset $S \subseteq Y$ such that $f(S) \subseteq S$ for all $f \in P$ satisfying $d(f) = c(f) = Y$. We say that P is *faithful* if given a nonidentity function $f \in P$ with $d(f) = c(f)$, then there is $x \in d(f)$ such that $f(x) \neq x$. By abuse of notation, we let P denote the category having the domains and codomains of functions in P as objects and the functions of P as morphisms. We let the obvious functor $P \rightarrow \text{Top}$ be denoted by t_P . Let G_P denote the opposite category of P and let $s_P : G_P \rightarrow \text{Top}^{\text{op}}$ denote the opposite functor of t_P . We will call s_P the *category dynamical system on X defined by the partially defined dynamical system P* .

Proposition 25. *If X is a topological space and P is a partially defined dynamical system on X , then P is topologically free (minimal, faithful) if and only if s_P is topologically free (minimal, faithful) as a category dynamical system.*

Proof. This follows immediately from the definition of topological freeness (minimality, faithfulness) of P and s_P . \square

To illustrate the above definitions and results, we end the article with some concrete examples of partially defined dynamical systems.

Example 26. Suppose that we let X denote the real numbers equipped with its usual topology and we let Y denote the set of nonnegative real numbers equipped with the relative topology induced by the topology on X . Let $\text{sqr} : X \rightarrow Y$ and $\text{sqrt} : Y \rightarrow X$ denote the square function and the square root function, respectively. Furthermore, let $\text{abs} : X \rightarrow X$ denote the absolute value. Let P be the partially defined dynamical system with $\text{ob}(P) = \{X, Y\}$ and $\text{mor}(P) = \{\text{id}_X, \text{id}_Y, \text{sqr}, \text{sqrt}, \text{abs}\}$. Then we get the following table of partial composition for P

\circ	id_X	id_Y	sqr	sqrt	abs
id_X	id_X	$*$	$*$	sqrt	abs
id_Y	$*$	id_Y	sqr	$*$	$*$
sqr	sqr	$*$	$*$	id_Y	sqr
sqrt	$*$	sqrt	abs	$*$	$*$
abs	abs	$*$	$*$	sqrt	abs

Put $G = P^{\text{op}}$ and let A_X and A_Y denote the set of continuous complex valued functions on X and Y respectively. Take $f_X, f'_X, g_X, g'_X, h_X, h'_X \in A_X$ and $f_Y, f'_Y, g_Y, g'_Y \in A_Y$. Then the product of

$$B_1 := f_X u_{\text{id}_X} + g_X u_{\text{abs}} + h_X u_{\text{sqr}} + f_Y u_{\text{id}_Y} + g_Y u_{\text{sqrt}}$$

and

$$B_2 := f'_X u_{\text{id}_X} + g'_X u_{\text{abs}} + h'_X u_{\text{sqr}} + f'_Y u_{\text{id}_Y} + g'_Y u_{\text{sqrt}}$$

3 Minimal and Faithful Category Dynamical Systems

in the skew category algebra $A \rtimes^\sigma G$ equals

$$\begin{aligned} B_1 B_2 = & f_X f'_X u_{\text{id}_X} + (f_X g'_X + g_X (f'_X \circ \text{abs}) + g_X (g'_X \circ \text{abs}) + h_X (g'_Y \circ \text{sqr})) u_{\text{abs}} + \\ & + (f_X h'_X + h_X (f'_Y \circ \text{sqr}) + g_X (h'_X \circ \text{abs})) u_{\text{sqr}} + (f_Y f'_Y + g_Y (h'_X \circ \text{sqr})) u_{\text{id}_Y} + \\ & + (f_Y g'_Y + g_Y (f'_X \circ \text{sqr}) + g_Y (g'_X \circ \text{sqr})) u_{\text{sqr}}. \end{aligned}$$

Now we examine the properties (i)-(iv) in Theorem 3.

Property (i) is false. Indeed, the ideal I of $A \rtimes^\sigma G$ generated by u_{abs} equals

$$A_X u_{\text{abs}} + A_X u_{\text{sqr}} + A_Y u_{\text{sqr}}.$$

Hence I is a nontrivial ideal of $A \rtimes^\sigma G$, which, in particular, implies that $A \rtimes^\sigma G$ is not simple.

By direct inspection of the table of partial composition for P it follows that property (ii) is false.

Property (iii) is false. In fact, if we let S be any subset of the set of the non-negative real numbers, then $\text{abs}(S) = S$. Hence P is not minimal.

Property (iv) is true. Indeed, the only nonidentity function in P that has equal domain and codomain is abs . But $\text{abs}(x) = -x \neq x$ for any negative real number.

Remark 27. It can be shown (see [11, Example 28] for the details) that the partially defined dynamical system in Example 26 is not topologically free.

Example 28. Let X denote a set equipped with the discrete topology. We now consider two partially defined dynamical systems P on X .

(a) Let P be the partially defined dynamical system on X having one-element subsets of X as objects and the unique functions between such sets as morphisms. Now we examine the properties (ii), (iii) and (iv) in Theorem 3. It is clear that P is a locally abelian inverse connected (small) groupoid so (ii) holds. By the definition of P it follows directly that it is both minimal and faithful. Therefore, by Theorem 3, we get that $A \rtimes^\sigma G$ is simple. We leave it as an exercise to the reader to show that $A \rtimes^\sigma G$ is isomorphic as a complex algebra to the direct limit $\varinjlim M_Y(\mathbb{C})$, where the direct limit is taken over finite subsets Y of X and we let $M_Y(\mathbb{C})$ denote the complex subalgebra of $A \rtimes^\sigma G$ generated by elements of the form u_m for $m \in \text{mor}(P)$ with $d(m), c(m) \in Y$. The maps $M_Y(\mathbb{C}) \rightarrow M_{Y'}(\mathbb{C})$, for finite subsets Y and Y' of X with $Y \subseteq Y'$, defining the direct limit, are defined by sending u_n , for $n \in P$ with $d(n), c(n) \in Y$, to u_n . If X is a finite set of cardinality n , then it is clear that $\varinjlim M_Y(\mathbb{C})$ is isomorphic to the ring $M_n(\mathbb{C})$ of $n \times n$ complex matrices. In particular, we now retrieve simplicity of $M_n(\mathbb{C})$.

(b) Let P be the partially defined dynamical system on X consisting of *all* subsets of X as objects and *all* maps between such sets as morphisms. Then P is a small category which is not a groupoid. Now we examine the properties (i)-(iv) in Theorem 3. It is easy to see that P is not topologically free. By Proposition 15 and Proposition 14 we conclude that $A \rtimes^\sigma G$ is not simple, so (i) is false. By choosing $e, f \in \text{ob}(G)$ (i.e. subsets of X) of different cardinality, it is easy to see that P is not inverse connected, so (ii) is false. It is clear that P is minimal and faithful, so (iii) and (iv) are true.

Acknowledgements

The second author was partially supported by The Swedish Research Council (postdoctoral fellowship no. 2010-918) and The Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation.

References

- [1] R. J. Archbold and J. S. Spielberg, Topologically free actions and ideals in discrete C^* -dynamical systems, *Proc. of Edinburgh Math. Soc.* **37** (1993), 119–124.
- [2] B. Blackadar, *Operator Algebras: Theory of C^* -algebras and von Neumann algebras*, Encyclopaedia of Mathematical Sciences, 122. Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006.
- [3] K. R. Davidson, *C^* -algebras by Example*, Memoirs of the American Mathematical Society, no. 75, American Mathematical Society, Providence RI, 1996.
- [4] E. G. Effros and F. Hahn, *Locally compact transformation groups and C^* -algebras*, Memoirs of the American Mathematical Society, no. 75, American Mathematical Society, Providence RI, 1967.
- [5] G. A. Elliott, Some simple C^* -algebras constructed as crossed products with discrete outer automorphism groups, *Publ. Res. Inst. Math. Sci.* **16** (1980), 299–311.
- [6] R. Exel and A. Vershik, C^* -algebras of irreversible dynamical systems, *Canad. J. Math.* **58**(1) (2006), 39–63.
- [7] S. Kawamura and J. Tomiyama, Properties of topological dynamical systems and corresponding C^* -algebras, *Tokyo. J. Math.* **13** (1990), 251–257.
- [8] A. Kishimoto, Outer automorphisms and reduced crossed products of simple C^* -algebras, *Comm. Math. Phys.* **81** (1981), 429–435.
- [9] G. Liu and F. Li, On Strongly Graded Rings and the Corresponding Clifford Theorem, *Algebra Colloq.* **13** (2006), 181–196.
- [10] P. Lundström, Separable Groupoid Rings, *Comm. Algebra* **34** (2006), 3029–3041.
- [11] P. Lundström and J. Öinert, Skew category algebras associated with partially defined dynamical systems, *Internat. J. Math.* **23**(4) (2012), pp. 16.
- [12] T. Masuda, Groupoid dynamical systems and crossed product. I. The case of W^* -systems. *Publ. Res. Inst. Math. Sci.* **20**(5) (1984), 929–957.
- [13] T. Masuda, Groupoid dynamical systems and crossed product. II. The case of C^* -systems. *Publ. Res. Inst. Math. Sci.* **20**(5) (1984), 959–970.

References

- [14] J. R. Munkres, *Topology*, Second Edition, Prentice-Hall, 2000.
- [15] F. J. Murray and J. von Neumann, On rings of operators, *Ann. of Math.* **37** (1936), 116–229.
- [16] F. J. Murray and J. von Neumann, On rings of operators IV, *Ann. of Math.* **44** (1943), 716–808.
- [17] J. von Neumann, Collected works vol. III. Rings of operators, Pergamon Press, 1961.
- [18] J. Öinert, Simple Group Graded Rings and Maximal Commutativity, Contemporary Mathematics (American Mathematical Society), Vol. 503, pp. 159–175 (2009).
- [19] J. Öinert and P. Lundström, Commutativity and Ideals in Category Crossed Products, *Proc. Est. Acad. Sci.* **59**(4) (2010), 338–346.
- [20] J. Öinert and P. Lundström, The Ideal Intersection Property for Groupoid Graded Rings, *Comm. Algebra* **40**(5) (2012), 1860–1871.
- [21] J. Öinert and P. Lundström, Miyashita Action in Strongly Groupoid Graded Rings, *Int. Electron. J. Algebra*, Vol. 11 (2012), 46–63.
- [22] J. Öinert, J. Richter and S. D. Silvestrov, Maximal commutative subalgebras and simplicity of Ore extensions. Preprint available at arXiv:1111.1292v1 [math.RA]
- [23] J. Öinert and S. D. Silvestrov, Commutativity and Ideals in Algebraic Crossed Products, *J. Gen. Lie T. Appl.* **2**, no. 4, 287–302 (2008).
- [24] J. Öinert and S. D. Silvestrov, On a Correspondence Between Ideals and Commutativity in Algebraic Crossed Products, *J. Gen. Lie T. Appl.* **2**, no. 3, 216–220 (2008).
- [25] J. Öinert and S. D. Silvestrov, Crossed Product-Like and Pre-Crystalline Graded Rings, Chapter 24 in 'Generalized Lie Theory in Mathematics, Physics and Beyond', Silvestrov, S.; Paal, E.; Abramov, V.; tolin, A. (Eds.), 16pp., Springer (2009).
- [26] J. Öinert, S. Silvestrov, T. Theohari-Apostolidi and H. Vavatsoulas, Commutativity and Ideals in Strongly Graded Rings, *Acta Appl. Math.* **108**, no. 3, 585–602 (2009).
- [27] J. Öinert, Simplicity of Skew Group Rings of Abelian Groups. Preprint available at arXiv:1111.7214v2 [math.RA]
- [28] A. L. T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Progress in Mathematics, 170. Birkhäuser Boston, Inc., Boston, MA, 1999.
- [29] G. K. Pedersen, *C*-algebras and their Automorphism Groups*, Academic Press, 1979.
- [30] S. C. Power, Simplicity of C*-algebras of minimal dynamical systems, *J. London Math. Soc.* **18** (1978), 534–538.

References

- [31] J. C. Quigg and J. S. Spielberg, Regularity and hyporegularity in C^* -dynamical system, *Houston J. Math.* **18** (1992), 139–152.
- [32] J. S. Spielberg, Free-product groups, Cuntz-Krieger algebras, and covariant maps, *Internat. J. Math.* **2** (1991), 457–476.
- [33] C. Svensson, S. Silvestrov and M. de Jeu, Dynamical Systems and Commutants in Crossed Products, *Internat. J. Math.* **18**(4) (2007), 455–471.
- [34] C. Svensson, S. Silvestrov and M. de Jeu, Connections between dynamical systems and crossed products of Banach algebras by \mathbb{Z} , in *Methods of Spectral Analysis in Mathematical Physics*, Operator Theory: Advances and Applications, Vol. 186, Birkhauser, 2009, pp. 391–402.
- [35] C. Svensson, S. Silvestrov and M. de Jeu, Dynamical Systems Associated with Crossed Products, *Acta Appl. Math.* **108** (2009), 547–559.
- [36] M. Takesaki, *Theory of operator algebras. II*, Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, 2003.
- [37] J. Tomiyama, Invitation to C^* -algebras and topological dynamics, World Sci., Singapore, New Jersey, Hong Kong, 1987.
- [38] J. Tomiyama, The interplay between topological dynamics and theory of C^* -algebras. Lecture Notes Series, 2. Global Anal. Research Center, Seoul, 1992.
- [39] D. P. Williams, Crossed Products of C^* -algebras, Mathematical Surveys and Monographs, American Mathematical Society, Providence, R. I., 2007.
- [40] G. Zeller-Meier, Produits croisés d’une C^* -algèbre par un groupe d’automorphismes, *J. Math. pures et appl.* **47** (1968), 101–239.